

Lecture 2 - Matrix Norms: Low rank Approximation

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1. Reminder

Definition (SVD): Let there be $M \in \mathbb{M}_{n \times m}(\mathbb{R})$. The SVD of this matrix is:

$$M = U \cdot \Sigma \cdot V^T$$

Where U, V are orthogonal. The columns $u_1, \dots, u_n$ of U are orthonormal basis for $\mathbb{R}^n$, and the columns $v_1, \dots, v_m$ of V are an orthonormal basis for $\mathbb{R}^m$.

Σ is a diagonal matrix where $\forall i \neq j, \Sigma_{ij} = 0$. Note that Σ is not necessarily square. We denote the values of the diagonal $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k : k = \min(m, n)$.

Definition trace:

$$\text{tr}(A) = \sum_{i=1}^n A_{i,i}$$

$$\text{tr}(AB) = \text{tr}(BA)$$

IEP: $E, F \subseteq \mathbb{R}^m$, then $\dim(E + F) = \dim(E) + \dim(F) - \dim(E \cap F)$ where $E + F = \text{span}(E \cup F)$

For $B : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\dim(\ker(B)) + \text{rk}(B) = m$

2. Matrix Norms

Definition (norm): A norm on a (real) vector space V is a function $\|\cdot\| : V \rightarrow [0, \infty)$ such that:

1. $\|x\| = 0 \iff x = 0 \quad \forall x \in V$
2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.
3. $\|\alpha x\| = |\alpha| \|x\| \quad \forall x \in V, \alpha \in \mathbb{R}$

2.1. Examples

For $V = \mathbb{R}^n$

$$\begin{aligned} \|x\| &= \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \\ 1 \leq p < \infty : \|x\|_p &= \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \\ \|x\|_\infty &= \max_{i \in \{1, \dots, n\}} |x_i| \end{aligned}$$

2.2. Matrix Norms

This is a fancy name where we take V to be the vector space of matrices: $V = \mathbb{M}_{n \times m}(\mathbb{R}) \cong \mathbb{R}^{n \cdot m}$. For example, for $M \in V$:

1. The operator norm / spectral norm is a measure of the size of a matrix, how much it can increase a vector v :

$$\|M\|_{\text{op}} = \max\{\|Mv\| : v \in \mathbb{R}^m, \|v\| = 1\}$$

Recall that $\|M\|_{\text{op}} = \sigma_1$, the largest singular value of M

2. Frobenius norm:

$$\|M\|_F = \left(\sum_{i=1}^n \sum_{j=1}^m M_{ij}^2 \right)^{\frac{1}{2}}$$

Claim: $\|M\|_F^2 = \text{tr}(M \cdot M^T)$

Proof:

$$\begin{aligned}(M \cdot M^T)_{ii} &= \sum_{j=1}^m \mathbb{M}_{ij} \cdot \mathbb{M}_{ji}^T \\ &= \sum_{j=1}^m \mathbb{M}_{ij}^2\end{aligned}$$

Claim: $\|M\|_F = \left(\sum_{i=1}^k \sigma_i^2 \right)^{\frac{1}{2}}$

Proof:

$$\begin{aligned}\|M\|_F^2 &= \text{tr}(MM^T) \\ &= \text{tr}\left(\underbrace{U\Sigma V^T}_M \cdot \underbrace{V\Sigma^T U^T}_{M^T}\right) \\ &= \text{tr}\left(\underbrace{U\Sigma\Sigma^T U^T}_A \underbrace{U^T}_B\right) \\ &= \text{tr}(U^T U \Sigma \Sigma^T) \\ &= \text{tr}(\Sigma \Sigma^T) \\ &= \|\Sigma\|_F^2 \\ &= \sum_{i=1}^k \sigma_i^2\end{aligned}$$

3. Low Rank Approximation

Definition $\text{rk}(M) = \text{rank}(M) = \dim(\text{Im}(M))$

Claim: Let $M \in \mathbb{M}_{n \times m}(\mathbb{R})$. The rank of M is the minimum number r such that

$$\exists A \in \mathbb{M}_{n \times r}(\mathbb{R}), B \in \mathbb{M}_{r \times m}(\mathbb{R}) : M = A \cdot B$$

It is clear that $\text{rk}(A) \leq r \wedge \text{rk}(A \cdot B) \leq \text{rk}(A) \leq r$. Therefore, if $M = A \cdot B \Rightarrow \text{rk}(M) \leq r$.

Claim: Suppose that the singular values of M are $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k \geq 0$. Then $\text{rk}(M)$ is the number of non-zero singular values.

$$\text{rk}(M) = \max\{i : i < k \wedge \sigma_i > 0\}$$

Proof:

$$\begin{aligned}M &= U\Sigma V^T \\ Mv_i &= \begin{cases} \sigma_i u_i & \text{if } 1 \leq i \leq k \\ 0 & \text{if } i > k \end{cases}\end{aligned}$$

Therefore, $\text{Im}(M)$ is spanned by $u_i : \sigma_i > 0$.

Question: Given $M \in \mathbb{M}_{n \times m}(\mathbb{R})$, find $B \in \mathbb{M}_{n \times m}(\mathbb{R})$ of rank $\leq r$ that minimises $\|M - B\|_{\text{op}}$ and $\|M - B\|_F$.

Definition: Given diagonal matrix Σ , and $r \leq k$, replace the last $k - r$ singular values on the diagonal with 0.

Theorem: For every matrix B of rank $\leq r$, it holds that $\|M - B\| \geq \|M - U\Sigma_{(r)}V^T\|$, for both the Frobenius, and the operator norm:

$$\begin{aligned}\|M - B\|_F &\geq \|M - U\Sigma_{(r)}V^T\|_F \\ &= \left(\sum_{i=r+1}^k \sigma_i^2 \right)^{\frac{1}{2}} \\ \|M - B\|_{\text{op}} &\geq \|M - U\Sigma_{(r)}V^T\|_{\text{op}} \\ &= \sigma_{r+1}\end{aligned}$$

Proof: This proof requires the Min-Max Theorem for singular values (below). Proof given below.

Theorem (Min-Max Theorem for singular values): Let $M \in \mathbb{M}_{n \times m}(\mathbb{R})$, with singular value decomposition $M = U\Sigma V^T$.

$$\forall 1 \leq i \leq k = \min(m, n)$$

$$\begin{aligned}\sigma_i &= \max_{\substack{E \subseteq \mathbb{R}^m \\ \dim(E)=i}} \left(\min_{\substack{v \in E \\ \|v\|=1}} \|Mv\| \right) \\ &= \min_{\substack{E \subseteq \mathbb{R}^m \\ \dim(E)=m-i+1}} \left(\max_{\substack{v \in E \\ \|v\|=1}} \|Mv\| \right)\end{aligned}$$

The second equality will not be proven, but is left as an exercise.

Proof (of first equality):

1. We need to show that there is $E \subseteq \mathbb{R}^m, \dim(E) = i : \min_{\substack{v \in E \\ \|v\|=1}} \|Mv\| = \sigma_i$
2. For every $E \subseteq \mathbb{R}^m, \dim(E) = i$, it holds that $\min_{\substack{v \in E \\ \|v\|=1}} \|Mv\| \leq \sigma_i$, that is $\exists v \in E, \|v\| = 1 : \|Mv\| \leq \sigma_i$
1. *Proof:* We will choose $E = \text{sp}\{v_1, \dots, v_i\} \Rightarrow \dim(E) = i$. Let $v \in E, \|v\| = 1$. Let us write

$$\begin{aligned}v &= \sum_{j=1}^i \alpha_j v_j \\ 1 = \|v\| &= \sum_{j=1}^i \alpha_j^2 \\ Mv &= \sum_{j=1}^i \alpha_j Mv_j \\ &= \sum_{j=1}^i \alpha_j \sigma_j u_j \\ \Rightarrow \|Mv\|^2 &= \sum_{j=1}^i \alpha_j^2 \sigma_j^2 \\ \|Mv\| &\geq \sigma_i \quad \forall v \in E, \|v\| = 1 \\ \|Mv\|^2 &= \sum_{j=1}^i \alpha_j^2 \sigma_j^2 \\ &\geq \sigma_i^2 \sum_{j=1}^i \alpha_j^2 \\ &= \sigma_i^2\end{aligned}$$

Choose $v = v_i : v \in E, \|v\| = 1 \wedge$

$$\begin{aligned}
\|Mv\| &= \|Mv_i\| \\
&= \|\sigma_i u_i\| \\
&= \sigma_i
\end{aligned}$$

2. *Proof:* Given E , we need to find the vector, rather than choosing E :

Let $E \subseteq \mathbb{R}^m$ of $\dim(E) = i$. Consider $F = \text{sp}\{v_i, v_{i+1}, \dots, v_m\}$, $\dim(F) = m - i + 1$. Therefore

$$\begin{aligned}
\dim(E) + \dim(F) &= m + 1 \\
\dim(E \cap F) &> 0
\end{aligned}$$

Let $v \in E \cap F$ of $\|v\| = 1$.

1. $v \in E$, $\|v\| = 1$
2. $v = \sum_{j=i}^m \alpha_j v_j$ since $v \in F$.

$$\begin{aligned}
1 = \|v\| &= \sum_{j=i}^m \alpha_j^2 \\
Mv &= \sum_{j=i}^m \alpha_j Mv_j \\
&= \sum_{j=i}^k \alpha_j \sigma_j u_j \\
\Rightarrow \|Mv\|^2 &= \sum_{j=i}^k \alpha_j^2 \sigma_j^2 \\
&\leq \sigma_i^2 \cdot \underbrace{\sum_{j=i}^k \alpha_j^2}_{\leq \|v\|^2 = 1} \\
&\leq \sigma_i^2
\end{aligned}$$

Theorem (Interlacing of singular values): For a matrix M_1 , we denote by $\sigma_i(M)$ the i -th singular value of M . $M, B \in \mathbb{M}_{n \times m}(\mathbb{R})$ and $\text{rk}(B) \leq r$. Then

$$\sigma_{i-r}(M) \underset{i > r}{\geq} \sigma_i(M + B) \geq \sigma_{i+r}(M)$$

Proof (of second inequality, uses Min-Max): We will construct $E \subseteq \mathbb{R}^m$, $\dim(E) = i$ such that

$$\forall v \in E, \|v\| = 1 : \|(M + B)v\| \geq \sigma_{i+r}(M)$$

Note that $\dim\left(\underbrace{\ker(B)}_{\{v: Bv=0\}}\right) \geq m - r$. Let us take

$$\begin{aligned}
F &= \text{sp}\{v_1, \dots, v_{i+r}\} \\
E &= F \cap \ker(B) \\
\dim(E) &= \dim(F \cap \ker(B)) \\
&\geq \dim(F) + \dim(\ker(B)) - m \\
&\geq (i + r) + (m - r) - m \\
&= i
\end{aligned}$$

We may show $\forall v \in E : \|(M + B)v\| \geq \sigma_{i+r}(M)$

- $(M_B)v = Mv$ since $v \in E \subseteq \ker(B)$
- $\|Mv\| \geq \sigma_{i+r}$ since $v \in \text{sp}\{v_1, \dots, v_{i+r}\}$

Theorem: $M = U\Sigma V^T$. For every matrix B of rank $\leq r$, it holds that $\|M - B\| \geq \|M - U\Sigma_{(r)}V^T\|$, for both the Frobenius, and the operator norm:

$$\begin{aligned}\|M - B\|_F &\geq \|M - U\Sigma_{(r)}V^T\|_F \\ &= \left(\sum_{i=r+1}^k \sigma_i^2 \right)^{\frac{1}{2}} \\ \|M - B\|_{\text{op}} &\geq \|M - U\Sigma_{(r)}V^T\|_{\text{op}} \\ &= \sigma_{r+1}\end{aligned}$$

Both inequalities are sharp (meaning that we may find B such that they are equivalent, where $B = U\Sigma_r V^T$, and is left as an exercise)

Proof:

$$\begin{aligned}\sigma_i(M - B) &\geq \sigma_{i+r}(M) \\ \|M - B\|_{\text{op}} &= \sigma_1(M - B) \\ &\geq \sigma_{r+1}(M) \\ \|M - B\|_F^2 &= \sum_{i=1}^k \sigma_i(M - B)^2 \\ &\geq \sum_{i=1}^k \sigma_{i+r}(M)^2 \\ &= \sum_{i=r+1}^k \sigma_i^2\end{aligned}$$

4. Linear PCA

Let there be vectors $x_1, \dots, x_n \in \mathbb{R}^n$. Let there be linear maps $B : \mathbb{R}^n \rightarrow \mathbb{R}^r$, and $A : \mathbb{R}^r \rightarrow \mathbb{R}^n$. We want to find A, B , that minimise

$$\sum_{i=1}^m \|ABx_i - x_i\|$$

The best way to do this is to perform SVD on the data. Let

$$X = (x_1 \dots x_m) = U\Sigma V^T$$

I.e., create a matrix X , where column i is the vector x_i . Additionally, let $P = AB$, which is of rank $\leq r$. So

$$\sum_{i=1}^m \|ABx_i - x_i\| = \|X - PX\|_F^2$$

$$\text{Solution: } PX = U\Sigma_{(r)}V^T$$

$$PU\Sigma V^T = U\Sigma_{(r)}V^T$$

$$PU\Sigma = U\Sigma_{(r)}$$

$$U^T PU\Sigma = \Sigma_{(r)}$$

$$U^T ABUS = \Sigma_{(r)}$$

$$\Rightarrow A = (u_1 \dots u_r) \quad B = \begin{pmatrix} u_1^T \\ \vdots \\ u_r^T \end{pmatrix}$$

So

$$P = A \cdot B = (u_1 \dots u_r) \begin{pmatrix} u_1^T \\ \vdots \\ u_r^T \end{pmatrix}$$

which is the projection on $\text{sp}\{u_1, \dots, u_r\}$