Lecture 3

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1. Introduction

We will be speaking about Infi today! Hooray! Not Linear Algebra, but Infi! Praise be, and gird thy loins. It appears that this will be a recap of stuff we discussed in the bachelors degree courses, but I'm writing this paragraph at the start of the lecture, so that may yet change…

2. Limits, Sets, Continuity

Definition $\mathbb{R}^d$: This is the d-dimensional euclidean vector space over $\mathbb{R}$.

$$\mathbb{R}^d = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} : x_1, ..., x_d \in \mathbb{R} \right\}$$

We define the standard basis, which has elements

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, ..., \quad e_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Definition Euclidean norm:

$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}$$

Properties of $\|\cdot\|$:

- Positive definite: $\forall x \ \ \|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0$
- Triangle inequality: $\forall x, y \ \ \|x + y\| \leq \|x\| + \|y\|$
- Homogeneity: $\forall \alpha \in \mathbb{R}, x \in \mathbb{R}^d, \ \|\alpha x\| = |\alpha| \|x\|$

Definition (distance): The *distance* between $x, y \in \mathbb{R}^d$ is $d(x, y) = \|x - y\|$
Properties of $d(\cdot, \cdot)$:

- Positive definite: $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$
- Triangle inequality: $\forall x, y, z, \ d(x, z) \leq d(x, y) + d(y, z)$
- Symmetry :$\forall x, y \ \ d(x, y) = d(y, x)$

Definition (Open ball): The *open ball* of radius $r > 0$, centred at $x \in \mathbb{R}^d$ is the set

$$B_{r(x)} = \{y \in \mathbb{R}^d : d(x, y) < r\}$$

Definition (Closed ball): The *closed ball* of radius $r > 0$, centred at $x \in \mathbb{R}^d$ is the set

$$\overline{B_r}(x) = \{y \in \mathbb{R}^d : d(x, y) \leq r\}$$

2.1. Limits

Definition A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$ *converges* to $x \in \mathbb{R}^d$ if $\forall i \in [d], \ \lim_{n \to \infty} x_i^n = x_i$

Proposition: Let $\{x^n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$. It converges to $x \in \mathbb{R}^d$ **if and only if** $\lim_{n\to\infty} \|x^n - x\| = 0$. We will note that this is arguing an alternate definition to the one above.
Proof: Let us assume $\lim_{n_>\infty} x^n = x$:

$$\forall n,\ \|x^n - x\| = \sqrt{\sum_{i=1}^{d} (x_i^n - x_i)^2}$$
$$\underbrace{\longrightarrow}_{n\to\infty} 0$$

Which follows since all of squared, sum, and square root are continuous functions, and by definition $x_i^n - x_i$ tends to 0.
Let us assume that $\|x^n \to x\|$:

$$\sqrt{(x_i^n - x_i)^2} = |x_i^n - x_i|$$
$$\leq \sqrt{\sum_{i=1}^{d} \left(x_j^n - x_j\right)^2}$$
$$7 \underbrace{\longrightarrow}_{n\to\infty} 0$$

Definition (Subsequence): Let $\{x^n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$. A *subsequence* is any sequence $\{x^{n_k}\}_{k=1}^{\infty}$, where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence in $\mathbb{N}$.
In short, a subsequence is an infinitely large subset of the original sequence, selected in increasing order of indices. If we consider $\mathbb{N}$ to be our sequence, then sub-sequences could be the even numbers, the odd numbers, the prime numbers, randomly selected numbers in increasing order, and so on.

Proposition: Fix sequences $\{x^n\}_{n=1}^{\infty}, \{y^n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$ and $\{\alpha^n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, with respective limits $x, y \in \mathbb{R}^d,\ \alpha \in \mathbb{R}$. Then:

1. $\lim_{n\to\infty} (x^n + y^n) = x + y$
2. $\lim_{n\to\infty} \alpha_n x^n = \alpha x$
3. Every subsequence of $\{x^n\}$ converges to x

Proof:

1. For every n:

$$\|x^n + y^n - (x + y)\| \overset{\triangle}{\leq} \|x^n - x\| + \|y^n - y\|$$
$$\longrightarrow 0 + 0$$

2.2. Sets

Definition (closed sets): A set $A \subseteq \mathbb{R}^d$ is *closed* if for every convergent sequence $\{x^n\}_{n=1}^{\infty} \subseteq A$, their limit is also in A (i.e. $\lim_{n\to\infty} x^n \in A$).
Example (half spaces): Let $w \in \mathbb{R}^d,\ a \in \mathbb{R}$, and define $H = \left\{x \in \mathbb{R}^d : \langle x, w\rangle \geq a\right\}$ where $\langle x, w\rangle = \sum_{i=1}^{d} x_i w_i$. Suppose $\{x^n\}_{n=1}^{\infty} \subseteq H$ converges to x. We must prove that $x \in F$, i.e. $\langle x, w\rangle \geq a$:

$$\langle x, w\rangle = \sum_{i=1}^{d} x_i w_i$$
$$= \sum_{i=1}^{d} w_i \lim_{n\to\infty} x_i^n$$
$$= \lim_{n\to\infty} \underbrace{\sum_{i=1}^{d} w_i x_i^n}_{\langle x^n, w\rangle \geq a}$$
$$\geq a$$

Example (sphere): $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Suppose $\{x^n\}_{n=1}^{\infty} \subseteq S^{d-1}$ that converges to x.

$$\begin{aligned}
\|x\| &= \sqrt{\sum_{i\in[d]} x_i^2} \\
&= \sqrt{\sum_{i\in[d]} \left(\lim_{n\to\infty} x_i^n\right)^2} \\
&= \lim_{n\to\infty} \sqrt{\sum_{i\in[d]} (x_i^n)^2} \\
&= \lim_{n\to\infty} \|x^n\| \\
&= 1
\end{aligned}$$

Thus the sphere is a closed set.

Definition (open set): A set U is *open* if $\forall x \in U \;\; \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq U$.

Example (open half spaces): Let $w \in \mathbb{R}^d, a \in \mathbb{R}$, and define $H = \{x \in \mathbb{R}^d : \langle w, x\rangle > a\}$. We claim that H is open: Let $x \in H$. Fix $\varepsilon > 0$, and suppose that $y \in B_\varepsilon(x)$:

$$\begin{aligned}
\langle y, w\rangle &= \langle y - x + x, w\rangle \\
&= \langle y - x, w\rangle + \underbrace{\langle x, w\rangle}_{>a} \\
|\langle y - x, w\rangle| &\underbrace{\leq}_{\text{Cauchy-Schwarz}} \|y - x\|\|w\| \\
&< \varepsilon\|w\| \\
\implies \langle y - x, w\rangle + \langle x, w\rangle &> \langle x, w\rangle - \varepsilon\|w\| \\
\langle x, w\rangle - \varepsilon\|w\| &> a \\
\iff \varepsilon\|w\| &< \langle x, w\rangle - a \\
\iff \varepsilon &< \frac{\langle x, w\rangle - a}{\|w\|}
\end{aligned}$$

Definition (boundary): The *boundary* of a set $A \subseteq \mathbb{R}^d$ denoted ∂A is the set of all $x \in \mathbb{R}^d$ s.t.

$$\forall \varepsilon > 0 \;\; B_\varepsilon(x) \cap A \neq \varnothing \wedge B_\varepsilon(x) \cap A^c \neq \varnothing$$

Example: Let $B^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ (the unit ball). Then $\partial B^d = S^{d-1}$, i.e., the unit sphere:

- $S^{d-1} \subseteq \partial B^d$: Let $x \in S^{d-1}, \; \varepsilon > 0$. Then $x \in B_\varepsilon(x)$. Also, $\left(1 + \frac{\varepsilon}{2}\right)x \in B_\varepsilon(x) \cap \left(B^d\right)^c$
- $\partial B^d \subseteq S^{d-1}$: Let $x \in \partial B^d, \; \varepsilon > 0$. By definition $\exists x_\varepsilon^- \in B^d \cap B_\varepsilon(x) \wedge x_\varepsilon^+ \in \left(B^d\right)^c \cap B_\varepsilon(x)$.

$$\begin{aligned}
\|x\| &\overset{\triangle}{\leq} \|x_\varepsilon^- - x\| + \|x_\varepsilon^-\| \\
&\leq 1 + \varepsilon \\
\|x\| &\geq \|x_\varepsilon^+\| - \|x - x_\varepsilon^+\| \\
&\geq 1 - \varepsilon
\end{aligned}$$

Therefore $1 - \varepsilon \leq \|x\| \leq 1 + \varepsilon \;\; \forall \varepsilon$, so $\|x\| = 1 \Rightarrow x \in S^{d-1}$.

We have shown both directions of containment, and so the sets are equivalent.

Definition (bounded sets): A set $A \in \mathbb{R}^d$ is *bounded* if $\exists R > 0 : A \subseteq B_R^d \coloneqq \{x \in \mathbb{R}^d : \|x\| < R\}$. I.e., a set is bounded if there exists a ball that contains the entire set.

Definition (compact sets): A set $A \subseteq \mathbb{R}^d$ is *compact* if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ has a convergent subsequence whose limit is in A.

Examples

- The closed d dimensional ball $\overline{B}^d$
- $[0,1]$
- S^{d-1}

Non-Examples:

- B^d (open)
- $(0,1)$ (consider $\frac{1}{n} \to 0 \notin (0,1)$)

Theorem: A set $K \subseteq \mathbb{R}^d$ is compact **if and only if** it is closed, and bounded.
Proof:

- $\Longrightarrow$: we have assumed that K is compact, let us prove that it is closed. Let $\{x_n\}_{n=1}^{\infty} \subseteq K$ converging to $x \in \mathbb{R}^d$. We need to prove that $x \in K$. Since K is compact, $\exists \left\{x_{n_k}\right\}_{k=1}^{\infty}$ subsequence of $\{x_n\}$ that converges to $y \in k$. Since $\{x_n\}$ is convergent, so every subsequence converges to

$$\begin{aligned} \lim_{n\to\infty} x_n &= x \\ &\Rightarrow y = x \\ &\Rightarrow x \in K \end{aligned}$$

  If K were *not* bounded, then $\forall n \in \mathbb{N}\ \exists x_n \in K : \|x_n\| \geq n$. We claim that $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence. Indeed, if $\lim_{k\to\infty} x_{n_k} = x$, then $\lim_{k\to\infty} \left\|x_{n_k}\right\| = \|x\|$. However; $\lim_{k\to\infty} \left\|x_{n_k}\right\| \geq \lim_{k\to\infty} n_k = \infty$.
- $\Longleftarrow$: Suppose that K is closed and bounded. Let $\{x_n\} \subseteq K$ be a sequence. Since K is closed, it suffices to prove that $\{x_n\}$ has a convergent subsequence. We proceed by induction on d: The base case is Bolzano–Weierstrass. Let us suppose that $\{x^{n_k}\} : \forall i < d\ \left\{x_i^{n_k}\right\}$ converges. Then $\left\{x_d^{n_k}\right\}$ is a bounded sequence in $\mathbb{R}$, and by Bolzano-Weierstrass, has a convergent subsequence $\left\{x_d^{n_{k_l}}\right\}$, but then $\{x^{n_{k_l}}\}$ is a convergent subsequence of $\{x^n\}$.

2.3. Continuity

Definition Let $A \subseteq \mathbb{R}^d$ and $f : A \to \mathbb{R}^k$. f is continuous at $x \in A$ if one of the following holds:

1. For every $\{x_n\} \subseteq A : x_n \to x \in A$, $\lim_{n\to\infty} f(x_n) = f(x)$
2. $\forall \varepsilon > 0\ \exists \delta > 0 : \forall y \in B_\delta(x) \cap A, |f(y) - f(x)| < \varepsilon$

We say that f is continuous if it is continuous at every $x \in A$.

Notation: $C(A)$ = class of continuous functions from A to $\mathbb{R}$

Lemma: Fix $A \subseteq \mathbb{R}^d, f, g : A \to \mathbb{R}^k$ that are continuous at $x \in A$. Then:

1. $f + g$ is continuous at x
2. $f \cdot g$ is continuous at x
3. If $\forall y \in A, g(y) \neq 0 \Rightarrow \frac{f}{g}$ is continuous at x.

Note, all these operations are pointwise.
Proof:

1. Let $\{x_n\} \subseteq A$ with $x^n \to x$.

$$\begin{aligned} (f+g)(x^n) &= f(x^n) + g(x^n) \\ &\longrightarrow f(x) + g(x) \\ &= (f+g)(x) \end{aligned}$$

2. Left as an exercise for the reader
3. Left as an exercise for the reader

Proposition: $C(A)$ is closed to addition, and multiplication (i.e. it is an "algebra").
Proof: Follows immediately from the above lemma.

Proposition: If $f : A \to \mathbb{R}^k, g : \mathbb{R}^k \to \mathbb{R}^d$ are continuous, then $g \circ f : A \to \mathbb{R}^d$ is also continuous.
Proof: Left as an exercise for the reader.

Theorem: Let $K \subseteq \mathbb{R}^d$ be compact, and let $f \in C(K)$. Then f is bounded and attains both a maximum and a minimum.
Proof: We first prove that f is bounded from above. Let us assume towards contradiction that there is some $\{x_n\} \subseteq K$ s.t. $\forall n \;\; f(x_n) \geq n$. K is compact, so there is a convergent subsequence $\left\{x_{n_k}\right\}$ with limit $x \in K$. However, then $f\left(x_{n_k}\right) \underset{k\to\infty}{\rightrightarrows} f(x)$. On the other hand $f\left(x_{n_k}\right) \geq n_k$, and so does not converge, which is a contradiction.
Since f is bounded, the set $\{f(x) : x \in k\}$ has a supremum M. We must show that $\exists x \in K : f(x) = M$. For every $n \;\; \exists x_n \in K$ with $f(x_n) \geq M - \dfrac{1}{n}$. Since K is compact, there is a convergent subsequence $\left\{x_{n_k}\right\}$ with limit $x \in K$. Since f is continuous, $f(x) = \lim\limits_{k\to\infty} f\left(x_{n_k}\right) \begin{cases} \leq M \\ \geq \lim_{n\to\infty} M - \frac{1}{n} = M. \end{cases}$