

Tutorial 1

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1. Linear Recurrence

Definition (Linear Recurrence Relation): A **linear recurrence relation** is an equation of the form

$$X_{n+k} = \sum_{i=0}^{k-1} c_i x_{n+i}$$

Where $\{c_i\} \subseteq \mathbb{R}$, and $k \in \mathbb{N}$ is fixed. k is called “the order of the recurrence relation”.

Example (Fibonacci):

$$x_{n+2} = \sum_{i=0}^1 x_{n+i}$$

So, a solution to the recurrence relation is the sequence $\bar{a} = (a_i)_{i \in \mathbb{N}} : a_{n+k} = \sum_{i=0}^{k-1} c_{i-1} a_{n+i}$.

In order for this to be defined properly, we need some initial condition. Let us say that $(a_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. We will denote by S the set of all the solutions to linear recurrence relations. Then, S is a vector field. As is required by a vector field:

1. $0 \in S$
2. Closed under addition: If we suppose the two series $(a_i)_{i \in \mathbb{N}}$, $(b_i)_{i \in \mathbb{N}}$, and we define

$$((a) + (b))_n = a_n + b_n$$

So, to check if $a + b \in S$:

$$\begin{aligned} a_{n+k} &= \sum_{i=0}^{k-1} c_i a_{n+i} \\ b_{n+k} &= \sum_{i=0}^{k-1} c'_i a_{n+i} \\ a_{n+k} + b_{n+k} &= \sum_{i=0}^{k-1} c_i (a_{n+i} + b_{n+i}) \end{aligned}$$

As required.

3. Closed to scalar multiplication: $\lambda \in \mathbb{R}, a \in S, \lambda a \in S$: Trivial.

Since we have a vector field, it is natural to ask questions about its dimension. Let us define

$$\begin{aligned} \pi : S &\rightarrow \mathbb{R}^k \\ \pi((a_i)_{i \in \mathbb{N}}) &:= (a_1, \dots, a_k)^T \end{aligned}$$

We will claim that π is a linear isomorphism. In order to check this:

1. We need to check that it maintains linearity:

$$\pi(\alpha a + \beta b) = (\alpha a_1 + \beta b_1, \dots, \alpha a_k + \beta b_k)$$

As required.

2. Is it a one-to-one function:

$$a \neq b \implies \pi(a) \neq \pi(b)?$$

No. Let us assume by contradiction that $\exists a \neq b : \pi(a) = \pi(b)$. Therefore there exists some minimum m for which $a_m \neq b_m$, and therefore so to they will be different after applying π

3. “into”: $\forall r \in \mathbb{R}^k$, we want some $s_r \in S$ for which $\pi(s_r) = r$. This is equivalent to asking given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, and $f(x) = 2x$ if it is true that $f(\mathbb{R}) = \mathbb{R}$, i.e. $\forall r \in \mathbb{R} \exists w : r = 2w$. This is fairly

trivial, since the first k elements may be arbitrary, and are not restricted to anything. We are defining that a_{n+k} is based off the sum of the earlier elements.

In order for a linear isomorphism to exist between two spaces, they must have the same dimension. This means that the space with infinite sequences S has a relatively small dimension.

Additionally, since it's one to one, we have infinite solutions. We can now reconstruct the basis of this. If we take $\{e_i\}_{i=1}^k$ to be the standard basis of $\mathbb{R}^k$, then the collection $\{\pi^{-1}(e_i)\}$ is the basis of S .

2. Matrix form

Recall that $x_{n+k} = \sum_{i=0}^{k-1} c_i x_{n+i}$. We may also write this in the form of a matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_0 & c_1 & c_2 & \dots & c_{k-1} & \end{pmatrix}$$

$$X_n = (x_n, x_{n+1}, \dots, x_{n+k-1})$$

$$AX_n = \left(x_{n+1}, x_{n+2}, \dots, x_{n+k-1}, \underbrace{\sum_{i=0}^{k-1} c_i x_{n+i}}_{x_{n+k}} \right)$$

$$\begin{aligned} \Rightarrow X_{n+1} &= AX_n \\ \Rightarrow X_{n+2} &= AX_{n+1} \\ &= AAX_n \\ &= A^2 X_n \\ \Rightarrow X_n &= A^n X_0 \end{aligned}$$

Let us assume that A is diagonalisable: $\exists P, D \in M_k(\mathbb{R})$, s.t. D is diagonal, then $A = PDP^{-1}$. So,

$$x_{n+l} = A^l x_n = A^{l+n} x_0 = PD^{l+n} P^{-1} x_0$$

This neat feature means that the n -th power of a $k \times k$ diagonalisable matrix may be computed in $O(k^3 \log n)$.

3. Characteristic Polynomial

Definition (Characteristic Polynomial): The characteristic polynomial of matrix A $P_A(x) = \det(XI - A)$. If λ is a root of $P_A(X)$, then λ is an eigenvalue of A .

Note that eigenvalues are useful, since if A is diagonalisable, then D will be a diagonal matrix, where the eigenvalues are the main diagonal.

Claim: The characteristic polynomial of A is

$$p(x) = x^k - \sum_{i=0}^{k-1} c_i x^i$$

Proof: We will prove by induction. When $k = 2$, then $\det(xI - A) = \det \begin{pmatrix} x & -1 \\ -c_0 & x - c_1 \end{pmatrix} = x^2 - c_1 x - c_0$.

Let us now assume that $k > 2$. We will use the Laplace expansion on the first column:

$$\det(xI - A) = \det \begin{pmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & -1 \\ -c_0 & -c_1 & -c_2 & \dots & x - c_{k-1} \end{pmatrix}$$

$$\begin{aligned}
&= x \det \begin{pmatrix} x & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ -c_1 & -c_2 & \dots & x - c_{k-1} \end{pmatrix} + (-1)^k c_0 \det \begin{pmatrix} -1 & 0 & \dots & 0 & 0 \\ x & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x & -1 \end{pmatrix} \\
&= x \left(x^{k-1} - \sum_{i=1}^{k-1} c_i x^{i-1} \right) + (-1)^k c_0 (-1)^{k-1} \\
&= x^k - \sum_{i=0}^{k-1} c_i x^i
\end{aligned}$$

Claim: If λ is an eigenvalue of $p(x)$, then:

1. $v_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots, \lambda^{k-1})^T$ is the corresponding eigenvector
2. $\bar{\lambda} = (\lambda^i)_{i=0}^\infty$ is in S , as a solution to the recurrence relation $x_{n+k} = \sum_{i=0}^{k-1} c_i x_{n+i}$
3. $\bar{\lambda} = \pi^{-1}(v_\lambda)$, where π is the isomorphism from above

Proof: For 1, let us consider the vector Av_λ . For $1 \leq i \leq k-1$, the i -th entry of Av_λ is:

$$(Av_\lambda)_i = \lambda^i = \lambda(v_\lambda)_i$$

For $i = k$, since λ is a root, $0 = p(\lambda) = \lambda^k - \sum_{i=0}^{k-1} c_i \lambda^i$, and

$$(Av_\lambda)_k = \sum_{i=0}^{k-1} c_i \lambda^i = \lambda^k = \lambda(v_\lambda)_k$$

Therefore, $Av_\lambda = \lambda v_\lambda$ as required.

For 2, note that for $n \geq k$:

$$\begin{aligned}
\lambda^n &= \lambda^{n-k} \lambda^k \\
&= \lambda^{n-k} \left(\sum_{i=0}^{k-1} c_i \lambda^i \right) \\
&= \sum_{i=0}^{k-1} c_i \lambda^{n-k+i}
\end{aligned}$$

showing that $\bar{\lambda}$ is a solution. The third claim follows from the definition of π .

3.1. Returning to Fibonacci

Since we have a basis to S , we may conclude that given the eigenvalues, we may create a general formula for any value n . For example, Fibonacci:

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
p(x) &= x^2 - x - 1 \\
\Rightarrow \varphi &= \frac{1 + \sqrt{5}}{2} \quad \wedge \quad -\varphi^{-1} = \frac{1 - \sqrt{5}}{2}
\end{aligned}$$

Given 0, 1 as the initial condition, then the n -th element is given by:

$$F_n = \frac{\varphi^n - (-\varphi)^n}{\sqrt{5}}$$